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On the local convergence of an iterative approach for inverse singular value problems[☆]

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Abstract

The purpose of this paper is to provide the convergence theory for the iterative approach given by M.T. Chu [Numerical methods for inverse singular value problems, SIAM J. Numer. Anal. 29 (1992), pp. 885–903] in the context of solving inverse singular value problems. We provide a detailed convergence analysis and show that the ultimate rate of convergence is quadratic in the root sense. Numerical results which confirm our theory are presented. It is still an open issue to prove that the method is Q-quadratic convergent as claimed by M.T. Chu.

Keywords: Inverse problems; Singular values; Root-convergence rate; Newton method

1. Introduction

Inverse problems arise in many practical situations such as medical imaging, exploration geophysics, and nondestructive evaluation where some general properties, for instance matrices, are to be determined from known data, e.g. eigenvalues, singular values, some prescribed entries. We refer to Chu and Golub [2] and Xu [6] for a comprehensive survey on structured and unstructured inverse eigenvalue and inverse singular value problems.

In this paper we consider the inverse singular value problem which is formally defined as follows.

Problem ISVP: Given n real $m \times n$ matrices $\{A_i\}_{i=1}^n$, $m \geqslant n$ and n nonnegative real numbers $\sigma_1^* \geqslant \sigma_2^* \geqslant \cdots \geqslant \sigma_n^*$, find $\mathbf{c} \in \mathbb{R}^n$ such that the singular values of the matrix

$$A(\mathbf{c}) \equiv c_1 A_1 + c_2 A_2 + \cdots + c_n A_n, \tag{1}$$

are precisely $\sigma_1^*, \ldots, \sigma_n^*$.

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This problem was first proposed by Chu and in [1] two numerical methods for solving Problem ISVP are presented. We restrict our attention to the second method of [1] which generalizes an effective iterative process proposed originally by Friedland et al. [3] for solving inverse eigenvalue problems. In [1] it is shown that the iterative approach is a variation of the Newton method and some convergence theory is provided. However, several theoretical issues raised in [1] deserve further attention. Here we show that the proof of local quadratic convergence in the quotient sense given in [1, Theorem 4.2] missed a block of free parameters which might not be in the second order of accuracy and we demonstrate the criticality of this block by providing some numerical examples. In addition, it seems to us that it is not clear how to derive local quadratic convergence of the iterative method proceeding as in [1]. Our purpose is to fill this gap by laying down a detailed convergence analysis of the iterative approach. Our analysis reveals that the iterative method converges at least quadratically in the root sense. We recall that the definitions of root-convergence and quotient-convergence are as follows

Definition 1 (*Ortega and Rheinboldt [5, Chapter 9]*). Let $\{\mathbf{x}^k\}$ be any convergent sequence with limit $\mathbf{x}^*$. Then, the quantities

$$Q_p\{\mathbf{x}^k\} = \begin{cases} 0 & \text{if } \mathbf{x}^k = \mathbf{x}^* \text{ for all but finitely many } k, \\ \limsup\limits_{k\to\infty} \dfrac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|^p} & \text{if } \mathbf{x}^k \neq \mathbf{x}^* \text{ for all but finitely many } k, \\ \infty & \text{otherwise,} \end{cases} \tag{2}$$

defined for all $p \in [1, \infty)$, are the *quotient-convergence factors* of $\{\mathbf{x}^k\}$ with respect to the norm $\|\cdot\|$ on $\mathbb{R}^n$ and

$$\mathrm{O}_Q(\mathbf{x}^*) = \begin{cases} \infty & \text{if } Q_p\{\mathbf{x}^k\} = 0, \quad \forall p \in [1, \infty), \\ \inf\{p \in [1,\infty) \mid Q_p\{\mathbf{x}^k\} = +\infty\} & \text{otherwise,} \end{cases} \tag{3}$$

is the Q-order of $\{\mathbf{x}^k\}$ at x^*.

Definition 2 (*Ortega and Rheinboldt [5, Chapter 9]*). Let $\{\mathbf{x}^k\}$ be any convergent sequence with limit $\mathbf{x}^*$. Then the numbers

$$R_p\{\mathbf{x}^k\} = \begin{cases} \limsup\limits_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/k} & \text{if } p = 1, \\ \limsup\limits_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/p^k} & \text{if } p > 1, \end{cases} \tag{4}$$

are the root-convergence factors of $\{\mathbf{x}^k\}$. The quantity

$$\mathrm{O}_R(\mathbf{x}^*) = \begin{cases} \infty & \text{if } R_p\{\mathbf{x}^k\} = 0, \quad \forall p \in [1, \infty), \\ \inf\{p \in [1,\infty) \mid R_p\{\mathbf{x}^k\} = 1\} & \text{otherwise,} \end{cases} \tag{5}$$

is called the R-order of $\{\mathbf{x}^k\}$ at x^*.

As the root-convergence is a weaker notion of convergence than the quotient-convergence, our result does not contradict the claim of [1]. In fact, proving the stronger result as stated in [1] remains an open issue.

In Section 2 we review the formulation and theory of the iterative method given in [1]. In Section 3 we present our convergence analysis and in Section 4 we show that our results are confirmed by numerical experiments.

In what follows, $\|\cdot\|$ denotes the Euclidean vector norm or its corresponding induced matrix norm. For any vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}} \in \mathbb{R}^n$ we use $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ to denote the singular values of $A(\mathbf{c})$ defined by (1), where $\sigma_1(\mathbf{c}) \geqslant \sigma_2(\mathbf{c}) \geqslant \cdots \geqslant \sigma_n(\mathbf{c}) \geqslant 0$. Assume that all the given singular values $\{\sigma_i^*\}_{i=1}^n$ are positive and distinct, and let $\Sigma_* = \mathrm{diag}(\sigma_1^*, \ldots, \sigma_n^*) \in \mathbb{R}^{m\times n}$, and $\mathcal{O}(n)$ denote the set of all orthogonal matrices in $\mathbb{R}^{n\times n}$. Finally, let $\|\cdot\|_F$ denote the Frobenius norm or the induced Frobenius norm in $\mathbb{R}^{m\times m} \times \mathbb{R}^{n\times n}$, see [1].

2. The iterative approach

In this section, we briefly recall the second method given in [1]. Define the affine subspace $\mathscr{A} \equiv \{A(\mathbf{c}) | \mathbf{c} \in \mathbb{R}^n\}$ and the surface $\mathscr{M}_s(\Sigma_*) \equiv \{U\Sigma_* V^T | U \in \mathcal{O}(m), V \in \mathcal{O}(n)\}$, i.e. the set of all matrices in $\mathbb{R}^{m\times n}$ with singular values $\sigma_1^* > \sigma_2^* \cdots > \sigma_n^* > 0$. Thus, solving Problem ISVP is equivalent to finding an intersection of $\mathscr{M}_s(\Sigma_*)$ and $\mathscr{A}$. The second method of [1] can be viewed as a variation of the Newton method where each iteration is composed of two major steps.

Let $\mathbf{c}^k$ be the current iterate and X_k a "lift" of $A(\mathbf{c}^k)$ from the affine subspace $\mathscr{A}$ to the surface $\mathscr{M}_s(\Sigma_*)$. In the first step, the new iterate $\mathbf{c}^{k+1}$ is computed so that $A(\mathbf{c}^{k+1})$ is an $\mathscr{A}$-intercept of a line that is tangent to the manifold $\mathscr{M}_s(\Sigma_*)$ at X_k. This amounts to finding two skew-symmetric matrices $F_{k+1} \in \mathbb{R}^{m\times m}$, $T_{k+1} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{c}^{k+1} \in \mathbb{R}^n$ such that

$$X_k + F_{k+1}X_k - X_k T_{k+1} = A(\mathbf{c}^{k+1}). \tag{6}$$

Notice that $X_k \in \mathscr{M}_s(\Sigma_*)$ implies that there exist $U_k \in \mathcal{O}(m)$ and $V_k \in \mathcal{O}(n)$ such that $U_k^T X_k V_k = \Sigma_*$. It follows from (6) that

$$\Sigma_* + H_{k+1}\Sigma_* - \Sigma_* K_{k+1} = U_k^T A(\mathbf{c}^{k+1}) V_k, \tag{7}$$

where $H_{k+1} = U_k^T F_{k+1} U_k \in \mathbb{R}^{m\times m}$ and $K_{k+1} = V_k^T T_{k+1} V_k \in \mathbb{R}^{n\times n}$ are skew-symmetric matrices.

In the second step, the matrix $A(\mathbf{c}^{k+1}) \in \mathscr{A}$ is lifted up to a new point $X_{k+1} \in \mathscr{M}_s(\Sigma_*)$ which is defined as

$$X_{k+1} \equiv U_{k+1}\Sigma_* V_{k+1}^T,$$

where U_{k+1} and V_{k+1} are two orthogonal matrices defined by

$$U_{k+1} = U_k R_{k+1} \quad \text{and} \quad V_{k+1} = V_k S_{k+1}. \tag{8}$$

Here, R_{k+1} and S_{k+1} are the Cayley transforms

$$R_{k+1} \equiv (I + \tfrac{1}{2}H_{k+1})(I - \tfrac{1}{2}H_{k+1})^{-1} \quad \text{and} \quad S_{k+1} \equiv (I + \tfrac{1}{2}K_{k+1})(I - \tfrac{1}{2}K_{k+1})^{-1}. \tag{9}$$

Overall we have
Iterative Algorithm:

(1) Given $\mathbf{c}^0$, compute the singular value $\{\sigma_i(\mathbf{c}^0)\}_{i=1}^n$, the normalized left singular vectors $\{\mathbf{u}_i(\mathbf{c}^0)\}_{i=1}^m$ and the normalized right singular vectors $\{\mathbf{v}_i(\mathbf{c}^0)\}_{i=1}^n$ of $A(\mathbf{c}^0)$, respectively. Let $U_0 = [\mathbf{u}_1^0, \ldots, \mathbf{u}_m^0] = [\mathbf{u}_1(\mathbf{c}^0), \ldots, \mathbf{u}_m(\mathbf{c}^0)]$, $V_0 = [\mathbf{v}_1^0, \ldots, \mathbf{v}_n^0] = [\mathbf{v}_1(\mathbf{c}^0), \ldots, \mathbf{v}_n(\mathbf{c}^0)]$, and

$$\boldsymbol{\sigma}^0 = (\sigma_1(\mathbf{c}^0), \ldots, \sigma_n(\mathbf{c}^0))^T.$$

(2) For $k = 0, 1, 2, \ldots,$ until convergence, do
 (a) Form the approximate Jacobian matrix J_k by

$$[J_k]_{ij} \equiv (\mathbf{u}_i^k)^T A_j \mathbf{v}_i^k, \quad 1 \leqslant i, j \leqslant n. \tag{10}$$

 (b) Solve $\mathbf{c}^{k+1}$ from the approximate Jacobian equation

$$J_k \mathbf{c}^{k+1} = \boldsymbol{\sigma}^*, \quad \boldsymbol{\sigma}^* = (\sigma_1^*, \ldots, \sigma_n^*)^T. \tag{11}$$

 (c) Form the matrix $A(\mathbf{c}^{k+1})$ by (1).
 (d) Form the matrix $W_k \equiv U_k^T A(\mathbf{c}^{k+1}) V_k$.
 (e) Compute the skew-symmetric matrices H_{k+1} and K_{k+1} by

$$[H_{k+1}]_{ij} = 0 \quad \text{for } n+1 \leqslant i \neq j \leqslant m, \tag{12}$$

$$[H_{k+1}]_{ij} = -[H_{k+1}]_{ji} = \frac{[W_k]_{ij}}{\sigma_j^*} \quad \text{for } n+1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n,$$

$$[H_{k+1}]_{ij} = -[H_{k+1}]_{ji} = \frac{\sigma_i^*[W_k]_{ji} + \sigma_j^*[W_k]_{ij}}{(\sigma_j^*)^2 - (\sigma_i^*)^2} \quad \text{for } 1 \leqslant i < j \leqslant n,$$

$$[K_{k+1}]_{ij} = -[K_{k+1}]_{ji} = \frac{\sigma_i^*[W_k]_{ij} + \sigma_j^*[W_k]_{ji}}{(\sigma_j^*)^2 - (\sigma_i^*)^2} \quad \text{for } 1 \leqslant i < j \leqslant n.$$

(f) Compute $U_{k+1} = [\mathbf{u}_1^{k+1}, \ldots, \mathbf{u}_m^{k+1}]$ and $V_{k+1} = [\mathbf{v}_1^{k+1}, \ldots, \mathbf{v}_n^{k+1}]$ by solving

$$(I + \tfrac{1}{2}H_{k+1})U_{k+1}^{\mathrm{T}} = (I - \tfrac{1}{2}H_{k+1})U_k^{\mathrm{T}},$$
$$(I + \tfrac{1}{2}K_{k+1})V_{k+1}^{\mathrm{T}} = (I - \tfrac{1}{2}K_{k+1})V_k^{\mathrm{T}}.$$

Clearly, equating the "diagonal" equations of (7) gives rise to (11). The skew-symmetric matrices H_{k+1} and K_{k+1} are obtained by the "off-diagonal" equations in (7). In such equations, the entries $[H_{k+1}]_{ij}$ for $n+1 \leqslant i \neq j \leqslant m$ are not bound to any equations at all. In principle, we can set these free parameters to be any values. In [1], the $((m-n)(m-n-1))/2$ unknowns located at the lower-right corner of H_{k+1} are set identically zeros. In fact, different allocations of the free parameters have an impact on the rate of convergence of the iterative algorithm and we will observe this fact in Section 4.

The convergence behavior of this iterative method was studied in [1]. Suppose that the ISVP has a solution $\mathbf{c}^*$ and that $A(\mathbf{c}^*) = U_*\Sigma_*V_*^{\mathrm{T}}$ with $U_* \in \mathcal{O}(m)$ and $V_* \in \mathcal{O}(n)$. Let $E_k \equiv (E_1^k, E_2^k) = (U_k - U_*, V_k - V_*)$ denote the error matrix at kth iteration. Then the following result states that the method is locally quadratically convergent in the quotient sense.

Theorem 1 (*Chu [1, Theorem 4.2]*). *Suppose that all singular values $\sigma_1^*, \ldots, \sigma_n^*$ are positive and distinct. Suppose also that the matrix $J^{(k)}$ defined in* (10) *is nonsingular. Then we have*

$$\|E_{k+1}\|_F = \mathrm{O}(\|E_k\|_F^2) \quad \text{and} \quad \|\mathbf{c}^{k+1} - \mathbf{c}^*\| = \mathrm{O}(\|E_k\|_F^2).$$

In [1], this theorem was proved as follows. Let

$$U_k^{\mathrm{T}} A(\mathbf{c}^*) V_k \equiv \mathrm{e}^{\hat{M}_k} \Sigma_* \mathrm{e}^{-\hat{N}_k}, \tag{13}$$

where $\mathrm{e}^{\hat{M}_k} = U_k^{\mathrm{T}} U_*$ and $\mathrm{e}^{\hat{N}_k} = V_k^{\mathrm{T}} V_*$. By [1, Lemma 4.1]

$$\|(\hat{M}_k, \hat{N}_k)\|_F = \mathrm{O}(\|E_k\|_F). \tag{14}$$

Together with (13), it follows that

$$U_k^{\mathrm{T}} A(\mathbf{c}^*) V_k = \Sigma_* + \hat{M}_k \Sigma_* - \Sigma_* \hat{N}_k + \mathrm{O}(\|E_k\|_F^2). \tag{15}$$

By taking the difference between (15) and (7), we get

$$U_k^{\mathrm{T}}(A(\mathbf{c}^*) - A(\mathbf{c}^{k+1}))V_k = (\hat{M}_k - H_{k+1})\Sigma_* - (\hat{N}_k - K_{k+1})\Sigma_* + \mathrm{O}(\|E_k\|_F^2). \tag{16}$$

The diagonal equations of (16) yields

$$J^{(k)}(\mathbf{c}^* - \mathbf{c}^{k+1}) = \mathrm{O}(\|E_k\|_F^2),$$

and from the nonsingularity of $J^{(k)}$, we have

$$\|\mathbf{c}^* - \mathbf{c}^{k+1}\| = \mathrm{O}(\|E_k\|_F^2).$$

Similarly, from the off-diagonal equations of (16) the following estimates are derived

$$\begin{aligned} &\|\hat{M}_k - H_{k+1}\|_F = \mathrm{O}(\|E_k\|_F^2), \\ &\|\hat{N}_k - K_{k+1}\|_F = \mathrm{O}(\|E_k\|_F^2). \end{aligned} \tag{17}$$

Because of (14), it must be that

$$\|(H_{k+1}, K_{k+1})\|_F = \mathrm{O}(\|E_k\|_F). \tag{18}$$

We observe that

$$\begin{aligned} E_1^{k+1} &\equiv U_{k+1} - U_* = U_k R_{k+1} - U_k \mathrm{e}^{\hat{M}_k} \\ &= U_k[(I + \tfrac{1}{2}H_{k+1}) - (I + \hat{M}_k + \mathrm{O}(\|\hat{M}_k\|^2))(I - \tfrac{1}{2}H_{k+1})](I - \tfrac{1}{2}H_{k+1})^{-1} \\ &= U_k[H_{k+1} - \hat{M}_k + \mathrm{O}(\|\hat{M}_k H_{k+1}\|) + \mathrm{O}(\|\hat{M}_k\|^2)](I - \tfrac{1}{2}H_{k+1})^{-1}. \end{aligned}$$

Thus, it is clear now that

$$\|E_1^{k+1}\| = \mathrm{O}(\|E_k\|_F^2).$$

A similar argument works for E_2^{k+1}. Therefore, the proof is completed.

We note that the estimate of $\|\hat{M}_k - H_{k+1}\|_F$ in (17) is incorrect. The reason is as follows. Since the system (16) shows that the $((m-n)(m-n-1))/2$ unknowns located at the lower-right corner of the matrix $\hat{M}_k - H_{k+1}$ are not bound to any equations at all, we cannot ensure that the first estimate in (17) holds. In fact, by (14) and (12), we have only

$$|[\hat{M}_k - H_{k+1}]_{ij}| = |[\hat{M}_k]_{ij}| = \mathrm{O}(\|E_k\|_F), \quad n+1 \leqslant i \neq j \leqslant m.$$

Thus as a whole, $\|\hat{M}_k - H_{k+1}\|_F = \mathrm{O}(\|E_k\|_F)$. Therefore, the quadratic convergence of the second method is not guaranteed when $m > n+1$.

In the next section, we develop the convergence analysis for the vector iterates $\{\mathbf{c}^k\}$ and the approximate singular vectors $\{U^{(k)}\}, \{V^{(k)}\}$ focusing on the convergence of the iterates $\{\mathbf{c}^k\}$.

3. Convergence analysis

In what follows, we assume that $\mathbf{c}^*$ is a solution of the ISVP and let $\mathbf{c}^k$ be the kth iterate produced by the iterative algorithm.

3.1. Preliminary lemmas

In this subsection, we give some preliminary lemmas, which are necessary for the convergence analysis. We first give the perturbation bound for singular values.

Lemma 1 (*Golub and Van Loan [4, Corollary 8.6.2]*). *If B and $B+E$ are in $\mathbb{R}^{m\times n}$ with $m \geqslant n$, then, for any $1 \leqslant k \leqslant n$,*

$$|\sigma_k(B+E) - \sigma_k(B)| \leqslant \|E\|,$$

where $\sigma_k(B)$ denotes the kth largest singular value of B.

In the following lemma, we give a perturbation bound for $A(\mathbf{c})$ defined in (1).

Lemma 2. *For any $\mathbf{c}, \bar{\mathbf{c}} \in \mathbb{R}^n$, we have*

$$\|A(\mathbf{c}) - A(\bar{\mathbf{c}})\| \leqslant \left(\sum_{i=1}^{n} \|A_i\|^2\right)^{1/2} \|\mathbf{c} - \bar{\mathbf{c}}\|, \tag{19}$$

where $A(\mathbf{c})$ is defined in (1).

Proof. The thesis follows from (1) noting that

$$\|A(\mathbf{c}) - A(\bar{\mathbf{c}})\| \leqslant \sum_{i=1}^{n} |c_i - \bar{c}_i| \|A_i\| \leqslant \left(\sum_{i=1}^{n} \|A_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} |c_i - \bar{c}_i|^2\right)^{1/2}. \qquad \square$$

Now, let the singular value decomposition of $A(\mathbf{c}^*)$ be

$$A(\mathbf{c}^*) = U_* \Sigma_* V_*^{\mathrm{T}}, \tag{20}$$

and define

$$J_* = [(J_*)_{ij}] \quad \text{with } (J_*)_{ij} = (\mathbf{u}_i^*)^{\mathrm{T}} A_j \mathbf{v}_i^*, \quad 1 \leqslant i, j \leqslant n,$$

where $\mathbf{u}_i^*$ and $\mathbf{v}_i^*$ are the ith column of U_* and V_*, respectively. In what follows we always assume that J_* is nonsingular. Thus, letting $U_* = [U_{*1}, U_{*2}]$ with $U_{*1} \in \mathbb{R}^{m\times n}$, by the continuity of the matrix inverse there exist positive numbers δ and C such that if $\mathbf{u}_i \in \mathbb{R}^m$, $\mathbf{v}_i \in \mathbb{R}^n$ satisfy

$$\max\{\|[\mathbf{u}_1, \dots, \mathbf{u}_n] - U_{*1}\|, \|[\mathbf{v}_1, \dots, \mathbf{v}_n] - V_*\|\} \leqslant \delta, \tag{21}$$

then the matrix $J = [\mathbf{u}_i^{\mathrm{T}} A_j \mathbf{v}_i]$ is nonsingular and

$$\|J^{-1}\| \leqslant C. \tag{22}$$

We then define

$$\gamma_1 = \tfrac{23}{2}\sigma_1^*, \quad \gamma_2 = \sqrt{n}\,\gamma_1 C, \quad \gamma_3 = \sqrt{2}\eta(\beta\gamma_2 + \gamma_1), \quad \gamma_4 = \sqrt{2n}\eta C\beta^2, \quad \eta = \frac{2n\sigma_1^*}{d_*} + \frac{1}{\sigma_n^*}, \tag{23}$$

where

$$\beta = \left(\sum_{i=1}^{n} \|A_i\|^2\right)^{1/2} \quad \text{and} \quad d_* = \min_{i\neq j} |\sigma_i^{*2} - \sigma_j^{*2}|. \tag{24}$$

Also, partition Σ_* as

$$\Sigma_* = \begin{bmatrix} \Sigma_{*1} \\ 0 \end{bmatrix}$$

with $\Sigma_{*1} \in \mathbb{R}^{n\times n}$ and U_k as $U_k = [U_{k1}, U_{k2}]$ with $U_{k1} \in \mathbb{R}^{m\times n}$ for $k = 0, 1, \dots$. Then, we have the following lemma.

Lemma 3. *If*

$$\max\{\|U_{01} - U_{*1}\|, \|V_0 - V_*\|\} \leqslant \frac{\delta}{4}, \tag{25}$$

$$\rho_1 \equiv \sqrt{\|H_1\|^2 + \|K_1\|^2} < \min\left\{1, \frac{1}{2\gamma_3}, \frac{3\delta}{8(\gamma_3 + 3)}\right\} \equiv \varepsilon_0, \tag{26}$$

then for any $k \geqslant 1$ *the iterates* $\{\mathbf{c}^k\}$, $\{H_k\}$, $\{K_k\}$ *and* $\{U_k^{\mathrm{T}} A(\mathbf{c}^k) V_k\}$ *generated by the iterative algorithm satisfy*

$$\|U_k^{\mathrm{T}} A(\mathbf{c}^k) V_k - \Sigma_*\| \leqslant \gamma_1 (\|H_k\|^2 + \|K_k\|^2), \tag{27}$$

$$\|\mathbf{c}^{k+1} - \mathbf{c}^k\| \leqslant \gamma_2 (\|H_k\|^2 + \|K_k\|^2), \tag{28}$$

$$\sqrt{\|H_{k+1}\|^2 + \|K_{k+1}\|^2} \leqslant \gamma_3 (\|H_k\|^2 + \|K_k\|^2), \tag{29}$$

$$\|U_{k+1} - U_k\| \leqslant 2\gamma_3 (\|H_k\|^2 + \|K_k\|^2), \tag{30}$$

$$\|V_{k+1} - V_k\| \leqslant 2\gamma_3 (\|H_k\|^2 + \|K_k\|^2). \tag{31}$$

Proof. See Appendix A. $\square$

Finally, we estimate the errors in $\{\mathbf{u}_i(\mathbf{c}^k)\}_{i=1}^n$ and $\{\mathbf{v}_i(\mathbf{c}^k)\}_{i=1}^n$ in terms of $\|\mathbf{c}^k - \mathbf{c}^*\|$.

Lemma 4. *Let the given singular values $\{\sigma_i^*\}_{i=1}^n$ be positive and distinct, and U_* and V_* denote associated matrices of the normalized left and normalized right singular vectors of $A(\mathbf{c}^*)$, respectively. Let the vectors $\mathbf{u}_i(\mathbf{c}^k)$ and $\mathbf{v}_i(\mathbf{c}^k)$ stand for the unit left and unit right singular vectors of $A(\mathbf{c}^k)$, respectively. Then there exist positive numbers ε_1 and κ such that, if $\|\mathbf{c}^k - \mathbf{c}^*\| \leqslant \varepsilon_1$, we have*

$$\|[\mathbf{u}_1(\mathbf{c}^k), \ldots, \mathbf{u}_n(\mathbf{c}^k)] - U_{*1}\| \leqslant \kappa \|\mathbf{c}^k - \mathbf{c}^*\|, \tag{32}$$

$$\|[\mathbf{v}_1(\mathbf{c}^k), \ldots, \mathbf{v}_n(\mathbf{c}^k)] - V_*\| \leqslant \kappa \|\mathbf{c}^k - \mathbf{c}^*\|. \tag{33}$$

Proof. It follows from the analyticity of a simple singular value and its corresponding left and right singular vectors. The proof of this lemma is similar to [6, p. 249]. Therefore, we omit the proof here. □

3.2. R-Convergence rate

In this subsection, we will show that the three sequences of the iterates $\{\mathbf{c}^k\}$, $\{U_k\}$ and $\{V_k\}$ generated by the iterative method are all at least quadratically convergent in the root sense. Next, we prove the main result of this paper.

Theorem 2. *Let the given singular values $\{\sigma_i^*\}_{i=1}^n$ be positive and distinct. Then there exist $\varepsilon > 0$, $\tilde{\mathbf{c}} \in \mathbb{R}^n$, $\tilde{U} \in \mathcal{O}(m)$ and $\tilde{V} \in \mathcal{O}(m)$ such that if $\|\mathbf{c}^0 - \mathbf{c}^*\| \leqslant \varepsilon$, the iterates $\{\mathbf{c}^k\}$, $\{U_k\}$, $\{V_k\}$, and $\{U_k^{\mathrm{T}} A(\mathbf{c}^k) V_k\}$ generated by the iterative algorithm converge to $\tilde{\mathbf{c}}$, $\tilde{U}$, $\tilde{V}$, and $\tilde{U}^{\mathrm{T}} A(\tilde{\mathbf{c}}) \tilde{V} = \Sigma_*$, respectively.*

Proof. By Lemmas 1 and 2, we have

$$\max_i |\sigma_i(\mathbf{c}^0) - \sigma_i^*| \leqslant \|A(\mathbf{c}^0) - A(\mathbf{c}^*)\| \leqslant \beta \|\mathbf{c}^0 - \mathbf{c}^*\|. \tag{34}$$

By Lemma 4, if $\|\mathbf{c}^0 - \mathbf{c}^*\| \leqslant \min\{\varepsilon_1, \delta/(4\kappa)\}$, then

$$\max\{\|U_{01} - U_{*1}\|, \|V_0 - V_*\|\} \leqslant \kappa \|\mathbf{c}^0 - \mathbf{c}^*\| \leqslant \delta/4, \tag{35}$$

where δ is given in (21). Thus by (21) and (22), we know that J_0 is nonsingular and $\|J_0^{-1}\| \leqslant C$. Note that

$$U_0^{\mathrm{T}} A(\mathbf{c}^0) V_0 = \Sigma_0 = \operatorname{diag}(\sigma_1(\mathbf{c}^0), \ldots, \sigma_n(\mathbf{c}^0)), \tag{36}$$

$$U_0^{\mathrm{T}} A(\mathbf{c}^1) V_0 = \Sigma_* + H_1 \Sigma_* - \Sigma_* K_1. \tag{37}$$

Taking the difference between (36) and (37) yields

$$U_0^{\mathrm{T}} (A(\mathbf{c}^1) - A(\mathbf{c}^0)) V_0 = \Sigma_* - \Sigma_0 + H_1 \Sigma_* - \Sigma_* K_1. \tag{38}$$

The diagonal equations of (38) give rise to

$$J_0(\mathbf{c}^1 - \mathbf{c}^0) = \boldsymbol{\sigma}^* - \boldsymbol{\sigma}^0,$$

and so, by (34), we have

$$\|\mathbf{c}^1 - \mathbf{c}^0\| \leqslant C\sqrt{n} \max_i |\sigma_i(\mathbf{c}^0) - \sigma_i^*| \leqslant \sqrt{n} C \beta \|\mathbf{c}^0 - \mathbf{c}^*\|. \tag{39}$$

Similarly to the proofs of (63) and (64), from (38), we can derive that

$$\max\{\|H_1\|, \|K_1\|\} \leqslant \eta \|A(\mathbf{c}^1) - A(\mathbf{c}^0)\|, \tag{40}$$

where η is defined in (23). By Lemma 2, it follows from (39) that

$$\|A(\mathbf{c}^1) - A(\mathbf{c}^0)\| \leqslant \beta \|\mathbf{c}^1 - \mathbf{c}^0\| \leqslant \sqrt{n} C \beta^2 \|\mathbf{c}^0 - \mathbf{c}^*\|. \tag{41}$$

Then, from (40) and (41), we get

$$\max\{\|H_1\|, \|K_1\|\} \leqslant \eta\sqrt{n}C\beta^2\|\mathbf{c}^0 - \mathbf{c}^*\|,$$

and so we have

$$\sqrt{\|H_1\|^2 + \|K_1\|^2} \leqslant \eta\sqrt{2n}C\beta^2\|\mathbf{c}^0 - \mathbf{c}^*\| = \gamma_4\|\mathbf{c}^0 - \mathbf{c}^*\|, \tag{42}$$

where γ_4 is defined in (23).

Now we let $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$. If $\|\mathbf{c}^0 - \mathbf{c}^*\| \leqslant \varepsilon$, where

$$\varepsilon < \min\left\{1, \varepsilon_1, \frac{\delta}{4\kappa}, \frac{\varepsilon_0}{\gamma_4}, \frac{1}{\gamma_4\gamma}\right\}, \tag{43}$$

then, from (35) and (42), we obtain

$$\max\{\|U_{01} - U_{*1}\|, \|V_0 - V_*\|\} \leqslant \frac{\delta}{4} \quad \text{and} \quad \rho_1 \equiv \sqrt{\|H_1\|^2 + \|K_1\|^2} < \varepsilon_0.$$

Thus by Lemma 3, we have, for any $k \geqslant 1$,

$$\begin{aligned} &\|\mathbf{c}^{k+1} - \mathbf{c}^k\| \leqslant \gamma\rho_k^2, \quad \rho_{k+1} \leqslant \gamma\rho_k^2, \quad \|U_k^{\mathrm{T}}A(\mathbf{c}^k)V_k - \Sigma_*\| \leqslant \gamma\rho_k^2, \\ &\|U_{k+1} - U_k\| \leqslant 2\gamma\rho_k^2, \quad \|V_{k+1} - V_k\| \leqslant 2\gamma\rho_k^2. \end{aligned} \tag{44}$$

where $\rho_k \equiv \sqrt{\|H_k\|^2 + \|K_k\|^2}$.

Let $\rho = \gamma\rho_1$, by (43) and (42), we know that $\rho < 1$. From (44) we have for each $k \geqslant 2$,

$$\begin{aligned} \|\mathbf{c}^k - \mathbf{c}^{k-1}\| &\leqslant \gamma\rho_{k-1}^2 \leqslant \gamma(\gamma\rho_{k-2}^2)^2 = \gamma^{1+2}\rho_{k-2}^{2^2} \\ &\leqslant \cdots \leqslant \gamma^{1+2+2^2+\cdots+2^{k-2}}\rho_1^{2^{k-1}} \\ &\leqslant (\gamma^{(1+2+2^2+\cdots+2^{k-2})/2^{k-1}}\rho_1)^{2^{k-1}} \\ &\leqslant (\gamma\rho_1)^{2^{k-1}} \leqslant \rho^{2^{k-1}}. \end{aligned}$$

Thus for any integer $m \geqslant 1$,

$$\begin{aligned} \|\mathbf{c}^{k+m} - \mathbf{c}^k\| &\leqslant \sum_{l=1}^{m} \|\mathbf{c}^{k+l} - \mathbf{c}^{k+l-1}\| \leqslant \sum_{l=1}^{m} \rho^{2^{k+l-1}} = \sum_{l=1}^{m}(\rho^{2^{k-1}})^{2^l} \\ &\leqslant \sum_{l=1}^{m}(\rho^{2^{k-1}})^l = \frac{\rho^{2^{k-1}} - (\rho^{2^{k-1}})^{m+1}}{1 - \rho^{2^{k-1}}}. \end{aligned} \tag{45}$$

This shows that $\{\mathbf{c}^k\}$ is a Cauchy sequence since $\rho < 1$. Therefore, there exists a $\tilde{\mathbf{c}} \in \mathbb{R}^n$ such that $\{\mathbf{c}^k\}$ converge to $\tilde{\mathbf{c}}$.

Similarly, from (44) we have, for any integer $m > 1$,

$$\max\{\|U_{k+m} - U_k\|, \|V_{k+m} - V_k\|\} \leqslant 2\frac{\rho^{2^{k-1}} - (\rho^{2^{k-1}})^{m+1}}{1 - \rho^{2^{k-1}}}. \tag{46}$$

It shows that $\{U_k\}$ and $\{V_k\}$ are both Cauchy sequences. Thus, there exist two matrices $\tilde{U} \in \mathcal{O}(m)$ and $\tilde{V} \in \mathcal{O}(n)$ such that $\{U_k\}$ and $\{V_k\}$ converge to $\tilde{U}$ and $\tilde{V}$, respectively. Finally, from (44), we have that $\{U_k^{\mathrm{T}}A(\mathbf{c}^k)V_k\}$ converge to $\tilde{U}^{\mathrm{T}}A(\tilde{\mathbf{c}})\tilde{V} = \Sigma_*$. $\square$

Remark. It is straightforward to note from (44) that the sequence $\{\rho_k\}$ converges to zero Q-quadratically. Finally, it is worthwhile to point out that $\tilde{\mathbf{c}}$ may not be equal to the solution $\mathbf{c}^*$.

We end this section by establishing quadratic convergence of our method in the root sense.

Theorem 3. *Under the same conditions as in Theorem 2, the three sequences of iterates $\{\mathbf{c}^k\}$, $\{U_k\}$ and $\{V_k\}$ generated by the iterative algorithm are all locally convergent with root-convergence rate at least equal to 2.*

Proof. By Theorem 2, we know that $\{\mathbf{c}^k\}$ is locally convergent with

$$\lim_{k\to\infty} \mathbf{c}^k = \tilde{\mathbf{c}}.$$

Since $\rho < 1$, letting $m \to \infty$ in (45), we have, for each $k \geqslant 1$,

$$\|\tilde{\mathbf{c}} - \mathbf{c}^k\| \leqslant \frac{\rho^{2^{k-1}}}{1-\rho^{2^{k-1}}} \leqslant \xi \rho^{2^{k-1}},$$

where $\xi = \frac{1}{1-\rho} > 1$. Moreover,

(1) If $p = 1$, then

$$R_1\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \tilde{\mathbf{c}}\|^{1/k} \leqslant \limsup_{k\to\infty} \xi^{1/k}(\rho^{1/2})^{2^k/k} = 0.$$

(2) If $1 < p < 2$, then

$$R_p\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \tilde{\mathbf{c}}\|^{1/p^k} \leqslant \limsup_{k\to\infty} \xi^{1/p^k}(\rho^{1/2})^{(2/p)^k} = 0.$$

(3) If $p = 2$, then

$$R_p\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \tilde{\mathbf{c}}\|^{1/2^k} \leqslant \limsup_{k\to\infty} \xi^{1/p^k}\rho^{1/2} = \rho^{1/2} < 1.$$

(4) If $p > 2$, then

$$R_p\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \tilde{\mathbf{c}}\|^{1/p^k} \leqslant \limsup_{k\to\infty} \xi^{1/p^k}(\rho^{1/2})^{(2/p)^k} = 1.$$

Therefore, the root-convergence factors $R_p\{\mathbf{c}^k\}$ of $\{\mathbf{c}^k\}$ defined in (4) is such that $R_p\{\mathbf{c}^k\} = 0$ for any $p \in [1, 2)$ and $R_p\{\mathbf{c}^k\} \leqslant 1$ for any $p \in [2, \infty)$. According to (5), $\mathrm{O}_R(\mathbf{c}^*) \geqslant 2$.

By similar arguments, we can prove that $\{U_k\}$ and $\{V_k\}$ converge to their limits $\tilde{U}$ and $\tilde{V}$ quadratically in the root sense. □

4. Numerical experiments

In this section, we report some numerical experiments to show the performance of the iterative algorithm. For demonstration purpose, we consider the case when $m = 7$ and $n = 4$. The tests were performed using `Matlab` 6.1 with machine precision 2.2×10^{-16}. All the basis matrices were generated randomly from a normal distribution with mean 0.0 and variance 1.0. To make sure that the ISVP under testing does have a solution, we first randomly generate a vector $\mathbf{c}^* \in \mathbb{R}^4$. Then, singular values of the corresponding matrix $A(\mathbf{c}^*)$ are used as the prescribed singular values. We perturb each entry of the vector $\mathbf{c}^*$ by a uniform distribution between -1 and 1 and use the perturbed vector as the initial guess $\mathbf{c}^0$ for the iterations. In our experiments, the iterations are stopped when

$$\|U_k^{\mathrm{T}} A(\mathbf{c}^k) V_k - \Sigma_*\|_F \leqslant 10^{-13}.$$

Table 1 includes $\mathbf{c}^*$, the initial guess $\mathbf{c}^0$ and the corresponding limit point $\tilde{\mathbf{c}}$ for three cases. Table 2 lists the errors between $\mathbf{c}^0$, $\mathbf{c}^*$ and $\tilde{\mathbf{c}}$. The number `It.` of performed iterations is 9, 5, 10, respectively. We can see from Table 2 that for Case (a) and Case (c), the limit point $\tilde{\mathbf{c}}$ of the iteration is not equal to the original vector $\mathbf{c}^*$ to which $\mathbf{c}^0$ is reasonably close. In particular, in Case (c), $\mathbf{c}^0$ is nearer to $\tilde{\mathbf{c}}$ than to $\mathbf{c}^*$ while Case (a) is the opposite. We point out that

Table 1
Initial and final values of $\mathbf{c}^k$

	Case (a)	Case (b)	Case (c)
c_1^*	$2.4467e+00$	$1.0987e+00$	$2.1995e+00$
c_2^*	$9.9836e-01$	$4.5028e-01$	$7.2577e-01$
c_3^*	$1.4491e+00$	$1.0739e+00$	$-7.0029e-01$
c_4^*	$-1.0565e+00$	$-7.5681e-01$	$2.0901e+00$
c_1^0	$2.6943e+00$	$1.6498e+00$	$2.4376e+00$
c_2^0	$1.4342e+00$	$7.6865e-01$	$7.6459e-01$
c_3^0	$2.4266e+00$	$1.6628e+00$	$2.3170e-01$
c_4^0	$-1.9655e-01$	$-5.1570e-01$	$2.5964e+00$
$\tilde{c}_1$	$3.1675e+00$	$1.0987e+00$	$1.9495e+00$
$\tilde{c}_2$	$1.5874e+00$	$4.5028e-01$	$1.0043e+00$
$\tilde{c}_3$	$2.8446e-01$	$1.0739e+00$	$-2.7139e-01$
$\tilde{c}_4$	$-3.5270e-01$	$-7.5681e-01$	$2.2095e+00$

Table 2
Errors between $\mathbf{c}^0$, $\mathbf{c}^*$ and $\tilde{\mathbf{c}}$

	$\|\mathbf{c}^0-\mathbf{c}^*\|$	$\|\mathbf{c}^0-\tilde{\mathbf{c}}\|$	$\|\mathbf{c}^*-\tilde{\mathbf{c}}\|$
Case (a)	$1.3951e+00$	$2.2047e+00$	$1.6487e+00$
Case (b)	$8.9994e-01$	$8.9994e-01$	$6.1294e-15$
Case (c)	$1.0877e+00$	$8.3577e-01$	$5.8163e-01$

Table 3
Singular values of $A(\mathbf{c}^k)$

$\boldsymbol{\sigma}^k$	Case (a)	Case (b)	Case (c)
σ_1^0	$2.0092e+01$	$1.2835e+01$	$1.0998e+01$
σ_2^0	$1.1860e+01$	$8.0329e+00$	$1.0673e+01$
σ_3^0	$7.5045e+00$	$4.8781e+00$	$7.3750e+00$
σ_4^0	$3.3831e+00$	$2.7887e+00$	$3.9632e+00$
σ_1^*	$1.5364e+01$	$8.6619e+00$	$9.8737e+00$
σ_2^*	$1.0882e+01$	$6.0069e+00$	$7.8588e+00$
σ_3^*	$5.8869e+00$	$3.5380e+00$	$6.7673e+00$
σ_4^*	$2.6861e+00$	$2.1041e+00$	$3.3554e+00$

this occurrence is in accordance with our convergence results and with the convergence features of iterative processes based on Newton method.

Clearly, the singular values of $A(\tilde{\mathbf{c}})$ agree with those of $A(\mathbf{c}^*)$. Table 3 displays the singular values of $A(\mathbf{c}^0)$ and those of $A(\tilde{\mathbf{c}})$. Furthermore, Table 4 displays the distance between $\boldsymbol{\sigma}(\mathbf{c}^k)=(\sigma_1(\mathbf{c}^k),\ldots,\sigma_n(\mathbf{c}^k))$ and $\boldsymbol{\sigma}^*$ measured in the 2-norm. Note that $\{\boldsymbol{\sigma}(\mathbf{c}^k)\}$ converges fast.

In order to further illustrate our theoretical results, in Table 5, we give the convergence history of the sequences $\{\rho_k\}\equiv\{\sqrt{\|H_k\|^2+\|K_k\|^2}\}$, $\{\mathbf{c}^k\}$, $\{U_k\}$ and $\{V_k\}$ for Case (a). Here, the limits $\tilde{\mathbf{c}}$, $\tilde{U}$ and $\tilde{V}$ are computed up to full precision. From Table 5, we can observe that the four sequences converge fast.

Table 4
Convergence history of $\{\boldsymbol{\sigma}(\mathbf{c}^k)\}$

It.	Case (a)	Case (b)	Case (c)
0	5.1399e + 00	4.8769e + 00	3.1501e + 00
1	1.8645e + 00	2.2626e − 01	3.0338e − 01
2	5.5573e + 00	3.2821e − 02	5.3643e − 01
3	2.1745e + 00	4.2234e − 04	1.4622e − 01
4	4.9857e − 01	1.8717e − 07	7.6712e − 01
5	2.7352e − 02	1.5397e − 14	4.4488e − 01
6	2.2679e − 03		3.3527e − 02
7	6.6781e − 06		3.0239e − 03
8	1.2317e − 10		9.3265e − 06
9	4.7830e − 15		2.7301e − 10
10			6.2804e − 15

Table 5
Convergence history of $\{\rho_k\}$, $\{\mathbf{c}^k\}$, $\{U_k\}$ and $\{V_k\}$ for Case (a)

It.	ρ_k	$\|\mathbf{c}^k - \tilde{\mathbf{c}}\|$	$\|U_k - \tilde{U}\|$	$\|V_k - \tilde{V}\|$
0	1.7029e + 00	2.2047e + 00	2.5985e + 00	9.5168e − 01
1	4.4140e + 00	2.3630e + 00	2.1818e + 00	1.3436e + 00
2	1.5427e + 00	1.7445e + 00	1.2882e + 00	6.8597e − 01
3	9.8196e − 01	6.9574e − 01	8.9750e − 01	2.8418e − 01
4	1.9614e − 01	1.5478e − 01	1.9838e − 01	1.4848e − 01
5	7.7722e − 02	6.0905e − 02	7.2257e − 02	3.7492e − 02
6	3.8347e − 03	2.7500e − 03	3.2571e − 03	2.0557e − 03
7	1.7031e − 05	1.3086e − 05	1.4488e − 05	8.9536e − 06
8	2.9797e − 10	2.2557e − 10	2.5370e − 10	1.5630e − 10
9	1.0989e − 15	5.3705e − 15	5.1940e − 15	3.7348e − 15

In order to demonstrate the criticality of the block of free parameters in Eq. (7), we consider the vector $\mathbf{c}^*$ as in Case (a). The initial guess $\mathbf{c}^0$ is obtained via perturbing $\mathbf{c}^*$ by a random vector with entries in $[-1, 1]$. In Tables 6 and 7 we list the numerical results for varying values of the block of free parameters by setting

$$[H_{k+1}]_{ij} = -[H_{k+1}]_{ji} = \tau \quad \text{for } n+1 \leqslant i < j \leqslant m$$

where $\tau = 0.01, 0.1, 0.5, 1.0$. We can observe from Tables 6 and 7 that the rate of convergence of the sequences $\{\mathbf{c}^k\}$ and $\{\boldsymbol{\sigma}(\mathbf{c}^k)\}$ slows for increasing values of τ.

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Appendix A

Before we show Lemma 3, we give some preliminary lemmas.

Lemma 5. *If $E \in \mathbb{R}^{n\times n}$ and $\|E\| < 1$, then $I - \frac{1}{2}E$ is nonsingular and*

$$\|(I + \tfrac{1}{2}E)(I - \tfrac{1}{2}E)^{-1} - (I + E)\| \leqslant \|E\|^2. \tag{47}$$

Table 6
Convergence history of $\{\mathbf{c}^k\}$ for varying τ

	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.5$	$\tau = 1.0$
It.	$\|\mathbf{c}^k - \tilde{\mathbf{c}}\|$	$\|\mathbf{c}^k - \tilde{\mathbf{c}}\|$	$\|\mathbf{c}^k - \tilde{\mathbf{c}}\|$	$\|\mathbf{c}^k - \tilde{\mathbf{c}}\|$
0	1.3598e + 00	1.2979e + 00	1.3681e + 00	1.3951e + 00
1	1.0661e − 01	3.7436e − 01	2.7539e − 01	1.1115e + 00
2	4.9443e − 03	1.3517e − 01	1.3577e − 01	1.1678e + 00
3	3.5940e − 06	4.5366e − 03	2.1800e − 02	1.2473e − 01
4	1.7510e − 10	5.6647e − 05	1.1962e − 03	1.1583e − 01
5	1.3568e − 14	3.2971e − 07	1.9027e − 04	4.9893e − 02
6		2.4337e − 09	3.0123e − 05	2.0168e − 02
7		1.8128e − 11	4.7492e − 06	8.2382e − 03
8		1.3549e − 13	7.4910e − 07	3.4887e − 03
9		2.5704e − 15	1.1827e − 07	1.5060e − 03
10			1.8676e − 08	6.5070e − 04
⋮			⋮	⋮
15			1.8320e − 12	9.3786e − 06
16			2.8592e − 13	4.0210e − 06
17			4.1981e − 14	1.7241e − 06
⋮				⋮
23				1.0683e − 08
24				4.5787e − 09
25				1.9623e − 09
26				8.4100e − 10
⋮				⋮
37				6.9713e − 14
38				2.6121e − 14
39				8.0367e − 15

Proof. Clearly $I - \frac{1}{2}E$ is nonsingular. It is easy to verify that

$$(I - \tfrac{1}{2}E)^{-1} = I + \tfrac{1}{2}E + \tfrac{1}{4}E^2(I - \tfrac{1}{2}E)^{-1},$$

and

$$(I + \tfrac{1}{2}E)(I - \tfrac{1}{2}E)^{-1} = I + E + \tfrac{1}{4}E^2(I + (I + \tfrac{1}{2}E)(I - \tfrac{1}{2}E)^{-1}). \tag{48}$$

Since $\|E\| < 1$ we have

$$\|(I - \tfrac{1}{2}E)^{-1}\| \leqslant \frac{1}{1 - 1/2\|E\|} < 2,$$

and by (48) the inequality (47) follows. □

Lemma 6. *Let $B \in \mathbb{R}^{n\times n}$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n\times n}$ with $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$. If the skew-symmetric matrices H and K satisfy that*

$$H\Sigma - \Sigma K = B, \tag{49}$$

then we have

$$H = Q \circ (B\Sigma + \Sigma B^{\mathrm{T}}), \quad \|H\| \leqslant \frac{2n\sigma_1}{d}\|B\|, \tag{50}$$

$$K = Q \circ (\Sigma B + B^{\mathrm{T}}\Sigma), \quad \|K\| \leqslant \frac{2n\sigma_1}{d}\|B\|, \tag{51}$$

Table 7
Convergence history of $\{\boldsymbol{\sigma}(\mathbf{c}^k)\}$ for varying τ

	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.5$	$\tau = 1.0$
It.	$\|\boldsymbol{\sigma}(\mathbf{c}^k) - \boldsymbol{\sigma}^*\|$	$\|\boldsymbol{\sigma}(\mathbf{c}^k) - \boldsymbol{\sigma}^*\|$	$\|\boldsymbol{\sigma}(\mathbf{c}^k) - \boldsymbol{\sigma}^*\|$	$\|\boldsymbol{\sigma}(\mathbf{c}^k) - \boldsymbol{\sigma}^*\|$
0	7.8233e + 00	7.5642e + 00	1.3268e + 00	5.1399e + 00
1	1.6885e − 01	5.5406e − 01	1.0690e + 00	1.8645e + 00
2	7.6371e − 03	2.1887e − 01	2.2937e − 01	1.8921e + 00
3	1.7274e − 05	1.2194e − 02	3.0287e − 02	4.2543e − 01
4	7.0969e − 10	9.8245e − 05	1.6264e − 03	1.2323e − 01
5	4.8558e − 14	5.9383e − 07	2.2896e − 04	5.4324e − 02
6		4.3857e − 09	3.6042e − 05	2.1842e − 02
7		3.2666e − 11	5.7003e − 06	8.6885e − 03
8		2.4408e − 13	9.0077e − 07	3.5924e − 03
9		7.9565e − 15	1.4222e − 07	1.5327e − 03
10			2.2452e − 08	6.6149e − 04
⋮			⋮	⋮
15			2.2063e − 12	9.6128e − 06
16			3.5445e − 13	4.1185e − 06
17			6.2187e − 14	1.7656e − 06
⋮				⋮
23				1.0945e − 08
24				4.6907e − 09
25				2.0103e − 09
26				8.6158e − 10
⋮				⋮
37				7.1675e − 14
38				2.3335e − 14
39				1.3448e − 14

where $\circ$ *denotes the Hadamard product*, $d = \min_{i \neq j} |\sigma_i^2 - \sigma_j^2|$, *and* $Q = [q_{ij}]$ *with*

$$q_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \dfrac{1}{\sigma_j^2 - \sigma_i^2} & \text{otherwise.} \end{cases}$$

Proof. Since $H^{\mathrm{T}} = -H$ and $K^{\mathrm{T}} = -K$, from (49) we have

$$-\Sigma H + K\Sigma = B^{\mathrm{T}}. \tag{52}$$

Eliminating the matrix H in (49) and (52) gives rise to

$$K\Sigma^2 - \Sigma^2 K = \Sigma B + B^{\mathrm{T}}\Sigma.$$

Equating the off-diagonal elements yields (51). Further, by (51) we have

$$\|K\|_\infty \leqslant \frac{1}{d}\|\Sigma B + B^{\mathrm{T}}\Sigma\|_\infty \leqslant \frac{1}{d}(\|\Sigma\|_\infty\|B\|_\infty + \|B^{\mathrm{T}}\|_\infty\|\Sigma\|_\infty) \leqslant \frac{\sigma_1}{d}(\|B\|_\infty + \|B^{\mathrm{T}}\|_\infty), \tag{53}$$

where $\|\cdot\|_\infty$ denotes the row sum norm. Recalling that $\|K\| \leqslant \sqrt{n}\|K\|_\infty$, $\|B\|_\infty \leqslant \sqrt{n}\|B\|$ and $\|B^{\mathrm{T}}\|_\infty \leqslant \sqrt{n}\|B^{\mathrm{T}}\| = \sqrt{n}\|B\|$, (51) follows. Similarly, we can prove (50). □

Now, we can establish Lemma 3.

Proof of Lemma 3. By (21) and (22), (25) implies that J_0 is nonsingular and $\|J_0^{-1}\| \leqslant C$. Thus, $\mathbf{c}_1$, U_1 and V_1 are well defined.

First we show that (27)–(31) hold for $k=1$. By (9) and Lemma 5, we have

$$\|U_1 - U_0\| = \|U_0(R_1 - I)\| = \|R_1 - I\| \leqslant 2\|H_1\| \leqslant 2\rho_1,$$
$$\|V_1 - V_0\| = \|V_0(S_1 - I)\| = \|S_1 - I\| \leqslant 2\|K_1\| \leqslant 2\rho_1,$$

Therefore, by the upper bound (26) on ρ_1 it follows that

$$\|U_{11} - U_{*1}\| \leqslant \|U_{11} - U_{01}\| + \|U_{01} - U_{*1}\| \leqslant \|U_1 - U_0\| + \|U_{01} - U_{*1}\| \leqslant 2\rho_1 + \frac{\delta}{4} < \frac{3}{4}\delta + \frac{1}{4}\delta = \delta.$$

Similarly, we have

$$\|V_1 - V_*\| \leqslant \delta.$$

Thus by (21) and (22), we know that J_1 is nonsingular and $\|J_1^{-1}\| \leqslant C$.

In order to prove (27), let

$$R_1 = I + H_1 + E_1 \quad \text{and} \quad S_1 = I + K_1 + F_1. \tag{54}$$

By (9) and Lemma 5 we have

$$\|E_1\| \leqslant \|H_1\|^2 \quad \text{and} \quad \|F_1\| \leqslant \|K_1\|^2. \tag{55}$$

Notice that from (7) and (8)

$$U_0^{\mathrm{T}} A(\mathbf{c}^1) V_0 = \Sigma_* + H_1\Sigma_* - \Sigma_* K_1, \quad U_1 = U_0 R_1 \quad \text{and} \quad V_1 = V_0 S_1.$$

Then, by (54), a short calculation gives rise to

$$U_1^{\mathrm{T}} A(\mathbf{c}^1) V_1 = \Sigma_* + G_1, \tag{56}$$

where

$$\begin{aligned} G_1 = {} & H_1(\Sigma_* - H_1\Sigma_* + \Sigma_* K_1)K_1 - H_1^2\Sigma_* - \Sigma_* K_1^2 + E_1^{\mathrm{T}}(\Sigma_* + H_1\Sigma_* - \Sigma_* K_1)(I + K_1) \\ & + (I - H_1 + E_1^{\mathrm{T}})(\Sigma_* + H_1\Sigma_* - \Sigma_* K_1)F_1. \end{aligned}$$

Using (55) and the assumption $\max\{\|H_1\|, \|K_1\|\} < 1$ we have

$$\begin{aligned} \|G_1\| & \leqslant 3\sigma_1^*\|H_1\|\|K_1\| + \sigma_1^*\|H_1\|^2 + \sigma_1^*\|K_1\|^2 + 6\sigma_1^*\|H_1\|^2 + 9\sigma_1^*\|K_1\|^2 \\ & \leqslant \tfrac{3}{2}\sigma_1^*(\|H_1\|^2 + \|K_1\|^2) + 10\sigma_1^*(\|H_1\|^2 + \|K_1\|^2) \\ & = \gamma_1(\|H_1\|^2 + \|K_1\|^2), \end{aligned} \tag{57}$$

where γ_1 is defined in (23). This shows that (27) is true for $k=1$.

Combining (56) with

$$U_1^{\mathrm{T}} A(\mathbf{c}^2) V_1 = \Sigma_* + H_2\Sigma_* - \Sigma_* K_2$$

yields

$$U_1^{\mathrm{T}}(A(\mathbf{c}^2) - A(\mathbf{c}^1))V_1 = H_2\Sigma_* - \Sigma_* K_2 - G_1. \tag{58}$$

The diagonal equations of (58) give rise to

$$J_1(\mathbf{c}^2 - \mathbf{c}^1) = \mathbf{g}_1,$$

where $\mathbf{g}_1$ is the diagonal vector of the matrix $-G_1$, and so we have

$$\|\mathbf{c}^2 - \mathbf{c}^1\| \leqslant C\|\mathbf{g}_1\| \leqslant C\sqrt{n}\|G_1\| \leqslant \gamma_2(\|H_1\|^2 + \|K_1\|^2), \tag{59}$$

where γ_2 is defined in (23). This shows that (28) holds for $k=1$. Let

$$Z \equiv U_1^{\mathrm{T}}(A(\mathbf{c}^2)-A(\mathbf{c}^1))V_1+G_1=\begin{array}{c}\begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix}\begin{array}{l} n \\ m-n \end{array}\end{array}$$

with $Z_{11}\in\mathbb{R}^{n\times n}$ and $Z_{21}\in\mathbb{R}^{(m-n)\times n}$. Noting that H_2 has the form

$$H_2=\begin{bmatrix} H_{11}^{(2)} & -H_{21}^{(2)^{\mathrm{T}}} \\ H_{21}^{(2)} & 0 \end{bmatrix}$$

with $H_{11}^{(2)}\in\mathbb{R}^{n\times n}$, from (58) we obtain

$$H_{11}^{(2)}\Sigma_{*1}-\Sigma_{*1}K_2=Z_{11}, \tag{60}$$

and

$$H_{21}^{(2)}\Sigma_{*1}=Z_{21}. \tag{61}$$

By Lemma 6 and (60) it follows that

$$\|H_{11}^{(2)}\|\leqslant\frac{2n\sigma_1^*}{d_*}\|Z_{11}\|, \tag{62}$$

$$\|K_2\|\leqslant\frac{2n\sigma_1^*}{d_*}\|Z_{11}\|. \tag{63}$$

On the other hand, by (61) we have

$$\|H_{21}^{(2)}\|\leqslant\frac{1}{\sigma_n^*}\|Z_{21}\|.$$

This, together with (62), yields

$$\begin{aligned}\|H_2\|&\leqslant\left\|\begin{bmatrix} H_{11}^{(2)} & 0 \\ 0 & 0 \end{bmatrix}\right\|+\left\|\begin{bmatrix} 0 & -H_{21}^{(2)^{\mathrm{T}}} \\ H_{21}^{(2)} & 0 \end{bmatrix}\right\| \\ &=\|H_{11}^{(2)}\|+\|H_{21}^{(2)}\|\leqslant\frac{2n\sigma_1^*}{d_*}\|Z_{11}\|+\frac{1}{\sigma_n^*}\|Z_{21}\|\leqslant\eta\|Z\|,\end{aligned} \tag{64}$$

where η is defined in (23).

By Lemma 2, (57) and (59) we get

$$\|Z\|\leqslant\beta\|\mathbf{c}^2-\mathbf{c}^1\|+\|G_1\|\leqslant(\beta\gamma_2+\gamma_1)(\|H_1\|^2+\|K_1\|^2), \tag{65}$$

where β is defined in (24). Combining (63) and (64) with (65) yields

$$\sqrt{\|H_2\|^2+\|K_2\|^2}\leqslant\gamma_3(\|H_1\|^2+\|K_1\|^2), \tag{66}$$

where γ_3 is defined in (23). This shows that (29) is true for $k=1$. Moreover, (66) implies that

$$\sqrt{\|H_2\|^2+\|K_2\|^2}\leqslant\sqrt{\|H_1\|^2+\|K_1\|^2}, \tag{67}$$

since we have assumed that $\sqrt{\|H_1\|^2+\|K_1\|^2}<1/\gamma_3$.

Thus, by Lemma 5, it follows from (67), (66) and (26) that

$$\|U_2-U_1\|=\|U_1(R_2-I)\|\leqslant 2\|H_2\|\leqslant 2\gamma_3\rho_1^2,$$

$$\|V_2-V_1\|=\|V_1(S_2-I)\|\leqslant 2\|K_2\|\leqslant 2\gamma_3\rho_1^2,$$

which shows that (30) and (31) are true for $k=1$.

Now we show that the inequalities (27)–(31) hold for the integer k, assuming that they are true for all positive integer less than or equal to $k-1$. From (66) and the induction assumption, we can easily derive that

$$\sqrt{\|H_k\|^2+\|K_k\|^2}\leqslant\sqrt{\|H_1\|^2+\|K_1\|^2}. \tag{68}$$

Similarly to the proof of (56), we can show that

$$U_k^{\mathrm{T}}A(\mathbf{c}^k)V_k=\Sigma_*+G_k, \tag{69}$$

where

$$\|G_k\|\leqslant\gamma_1(\|H_k\|^2+\|K_k\|^2). \tag{70}$$

By the induction assumptions we know that, for $j=2,3\ldots,k$,

$$\|U_{j1}-U_{j-1,1}\|\leqslant\|U_j-U_{j-1}\|\leqslant 2\gamma_3\rho_{j-1}^2,\quad \|V_j-V_{j-1}\|\leqslant 2\gamma_3\rho_{j-1}^2,\quad \rho_j\leqslant\gamma_3\rho_{j-1}^2,$$

where $\rho_j=\sqrt{\|H_j\|^2+\|K_j\|^2}$. By (26), we get $\gamma_3\rho_1<1/2$ and also $\gamma_3\rho_1<3/8\delta$. Thus, we have

$$\begin{aligned}\|V_k-V_*\|&\leqslant\sum_{j=1}^{k}\|V_j-V_{j-1}\|+\|V_0-V_*\|\leqslant\sum_{j=2}^{k}2\gamma_3\rho_{j-1}^2+2\rho_1+\frac{\delta}{4}\\&\leqslant\sum_{j=2}^{k}2(\gamma_3\rho_1)^{2^{j-1}}+2\rho_1+\frac{\delta}{4}\leqslant 2\cdot\frac{(\gamma_3\rho_1)^2}{1-(\gamma_3\rho_1)^2}+2\rho_1+\frac{\delta}{4}\\&\leqslant 2\cdot\frac{2}{3}\gamma_3\rho_1+2\rho_1+\frac{\delta}{4}\leqslant\frac{\delta}{2}+\frac{\delta}{4}+\frac{\delta}{4}=\delta.\end{aligned}$$

Similarly, we can prove that $\|U_{k1}-U_{*1}\|\leqslant\delta$. Thus, it follows from (21) and (22) that J_k is nonsingular and $\|J_k^{-1}\|\leqslant C$.

Combining (69) with

$$U_k^{\mathrm{T}}A(\mathbf{c}^{k+1})V_k=\Sigma_*+H_{k+1}\Sigma_*-\Sigma_*K_{k+1},$$

we have

$$U_k^{\mathrm{T}}(A(\mathbf{c}^{k+1})-A(\mathbf{c}^k))V_k=H_{k+1}\Sigma_*-\Sigma_*K_{k+1}+G_k. \tag{71}$$

From (71), proceeding as in the proof of (59) and (66), we have

$$\|\mathbf{c}^{k+1}-\mathbf{c}^k\|\leqslant\gamma_2(\|H_k\|^2+\|K_k\|^2), \tag{72}$$

and

$$\sqrt{\|H_{k+1}\|^2+\|K_{k+1}\|^2}\leqslant\gamma_3(\|H_k\|^2+\|K_k\|^2).$$

This, together with (68), gives rise to

$$\sqrt{\|H_{k+1}\|^2+\|K_{k+1}\|^2}\leqslant\sqrt{\|H_k\|^2+\|K_k\|^2}\leqslant\sqrt{\|H_1\|^2+\|K_1\|^2},$$

which implies $\|H_{k+1}\|\leqslant\sqrt{\|H_{k+1}\|^2+\|K_{k+1}\|^2}<1$. Thus, by Lemma 5 and (72), we get

$$\|U_{k+1}-U_k\|=\|U_k(R_{k+1}-I)\|=\|R_{k+1}-I\|\leqslant 2\|H_{k+1}\|\leqslant 2\gamma_3(\|H_k\|^2+\|K_k\|^2).$$

Similarly, we can prove that (31) holds.

Therefore, by mathematical induction principle, we have showed that the inequalities (27)–(31) hold for all positive integers.

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